Journal of Intelligent Learning Systems and Applications, 2012, 4, 127-134
http://dx.doi.org/10.4236/jilsa.2012.42012 Published Online May 2012 (http://www.SciRP.org/journal/jilsa)

Generalizations of Rough Functions in Topological Spaces by Using Pre-Open Sets

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Received January 25th, 2011; revised March 6th, 2012; accepted March 14th, 2012

ABSTRACT

Functions are a means to link or transport from a world to another world may be similarly or completely different from the other world. In this paper we addressed the issue of rough functions and the possibility of transfer it from the real line to the topological abstract view that can be applied to intelligent information systems. The rough function approach has not been studied much specially from a topological point of view. Here we developed a new type of topological generalizations of rough functions with reference to how it is used in medical applications. Considering that the function is in the original a relation can be based on a review of all circular functions from the perspective of relations. Accordingly, the dream that the generalizations of rough functions are transferred to all papers prior to a comprehensive computer application.

Keywords: Rough Sets; Rough Numbers; Approximation Spaces; Topological Spaces; Fuzzy Sets

1. Introduction

Rough set theory [1], is an extension of set theory for the study of intelligent systems characterized by inexact, uncertain or insufficient information. Moreover, this theory may serve as a new mathematical tool to soft computing besides fuzzy set theory [2-4], and has been successfully applied in machine learning, information sciences, expert systems, data reduction, and so on. Recently, lots of researchers are interested to generalize this theory in many fields of applications [5-7]. In classical rough set theory, partition or equivalence (indiscernibility) relation is an important and primitive concept. But, partition or equivalence relation is still restrictive for many applications. To study this issue, several interesting and meaningful generalizations to equivalence relation have been proposed in the past, such as tolerance relations [8], topological bases and subbases [9-12]. Particularly, some researchers have used coverings of the universe of discourse for establishing the generalized rough sets by coverings [13]. Others [14-16] combined fuzzy sets with rough sets in a fruitful way by defining rough fuzzy sets and fuzzy rough sets. Furthermore, another group has characterized a measure of roughness of a fuzzy set making use of the concept of rough fuzzy sets [17-19]. They also suggested some possible real world applications of these measures in pattern recognition and image analysis problems. Some results of these generalizations are obtained about rough sets and fuzzy sets in [20-22].

Topological ideas are present not only in almost all areas of today mathematics, for example biochemistry [23] information systems [24] and others for more fields of topology applications [25] and its related links. The subject of topology itself consists of several different branches such as point set topology, algebraic topology and differential topology which have relatively little in common this richness of applications and differentiate between branches of topology implied a difficult to give an accurate definition for topology. The topology concepts like continuity, irresoluteness, compactness, connectedness, convergence, denseness and others are as basic to mathematicians. The topology structure τ on a set X is a general tool for constructing the above concepts. This tool contains many classes of near open sets such as: regular open [26], semi open sets [27], pre-open sets [28], β-open sets [29] and b-open sets [30]. Many authors used the previous types of near open sets to introduce some types of near continuous functions such as: In [28] the concept of pre-continuous functions are introduced. In [31] the concept of α-continuous functions is introduced.

The pair of lower and upper approximation operators is just a pair of interior and closure operators of a topology [32-34]. In [35] the concept of rough functions is introduced. In [35,36] we found the definition of the rough real number. In this paper, we propose to give a further study on rough functions and to introduce some concepts based on rough functions. In the beginning we will study rough sets on the real line.

In Section 2, we will initiate the notion of rough real functions. The aim of Section 3 is to define and study the new notion of "topological pre-rough function". The main goal of Section 4 is to initiate and study the pre-approximations of a function as a relation. Finally, we aim in Section 5 to define an alternative description of the topological pre-rough functions and topological pre-rough continuity.

A topological space [36] is a pair (X,τ) consisting of a set X and family τ of subsets of X satisfying the following conditions:

1) $\varphi, X \in \tau$.
2) τ is closed under arbitrary union.
3) τ is closed under finite intersection.

The pair (X,τ) is called a topological space, the elements of X are called points of the space, the subsets of X belonging to are called open set in the space, and the complement of the subsets of X belonging to τ be called closed set in the space; the family τ of open subsets of X is also called a topology for X.

$\overline{A} = \bigcap\left\{F \subseteq X : A \subseteq F,\ F \in \tau^c\right\}$ is called τ-closure of a subset $A \subset X$.

Evidently, A is the smallest closed subset of X which contains A. Note that A is closed iff $A = \overline{A}$.

$A^\circ = \bigcup\left\{G \subseteq X : G \subseteq A,\ G \in \tau\right\}$ is called the τ-interior of a subset $A \subseteq X$.

Evidently, A° is the union of all open subsets of X which containing in A. Note that A is open iff $A = A^\circ$. And $b(A) = \overline{A} - A^\circ$ is called the τ-boundary of a subset $A \subseteq X$.

Let A be a subset of a topological space (X,τ). Let $\overline{A}$, A° and $b(A)$ be closure, interior, and boundary of A respectively. A is exact if $b(A) = \varphi$, otherwise A is rough. It is clear A is exact iff $\overline{A} = A^\circ$.

Definition 1.1: A subset A of a topological space (X,τ) is called pro-open if $A \subset int\left(cl(A)\right)$.

The family of all pre-open sets of X is denoted by $PO(X)$. The complement of pre-open set is preclosed set. The family of preclosed sets is denoted by $PC(X)$.

Definition 1.2: A function $f:(X,\tau) \to (Y,\sigma)$ is said to be pre-continuous if $f^{-1}(G) \in PO(X)$ for every $G \in \sigma$ [28].

2. Rough Functions on Real Line

Let R^+ be the set of non-negative real numbers, and let $Seq \subseteq R^+$ be a sequence of real numbers defined by $x_1, x_2, x_3, \cdots, x_n, \cdots$ such that $x_1 < x_2 < x_3 < \cdots < x_n < \cdots$. The sequence Seq defines the partition $\pi(Seq)$ of R^+ by $\pi(Seq) = \left\{0, (0,x_1), x_1, (x_1,x_2), x_2, \cdots, (x_i,x_{i+1}), \cdots\right\}$, where (x_i, x_{i+1}) denote open intervals R^+. The sequence Seq is called a categorization of R^+ and the ordered pair $A = \left(R^+, \pi^*(Seq)\right)$ is an approximation space, where $\pi^*(Seq)$ is the equivalence relation associated with $\pi(Seq)$.

Let $A = \left(R^+, \pi^*(Seq)\right)$ be an approximation space. By $Seq(x)$ in A we denote the block of the partition $\pi(Seq)$ containing x, in particular if $x \in Seq$, we have $Seq(x) = \left\{x\right\}$, $clSeq\left(x\right)$ is the closure of $Seq(x)$ with respect to the usual topology on R. Let $A = \left(R^+, \pi^*(Seq)\right)$ be an approximation space, by $Q(x)$ we denote the closed interval $[0,x]$ for $x \in R^+$. For any $x \in R^+$, the Seq-lower and the Seq-upper approximations of $Q(x)$ in the approximation space $A = \left(R^+, \pi^*(Seq)\right)$ are defined respectively by

$$Seq_*\left[Q(x)\right] = \left\{y \in R^+ : Seq(y) \subseteq Q(x)\right\}$$
$$Seq^*\left[Q(x)\right] = \left\{y \in R^+ : Seq(y) \cap Q(x) \neq \varphi\right\}$$

The approximations of the closed interval $Q(x) = [0,x]$ can be understood as the approximations of the real number x which are simply the ends of the interval $Seq(x)$. The number x is a rough number if $Seq_*\left[Q(x)\right] \neq Seq^*\left[Q(x)\right]$, otherwise it is an exact number.

***Example* 2.1:** Let R^+ be the set of all non-negative real numbers, and let $N = \{1,2,3,\cdots\}$ be the set of natural numbers to be a sequence in R^+. Then the partition induced by N is

$\pi(N) = \left\{0, (0,1), 1, (1,2), 2, \cdots, n, (n,n+1), n+1, \cdots\right\}$

and hence, $A = \left(R^+, \pi^*(N)\right)$ is an approximation space. Also, for any number $x \in N$, we have $N(x) = \{x\}$ and for any $x \notin N$, $N(x) = \left(x_i, x_{i+1}\right)$ and $x \in \left(x_i, x_{i+1}\right)$, Then every number $x \notin N$ is a rough number in A.

According to Example 1, we followed the following steps to get the approximations of a number x, say $x = 1.2$. We remark that the required approximations of $x = 1.2$ can be obtained directly in one step by $Bnd\left(Q(1.2)\right) = N\left(1.2\right)$.

Let X and Y be two subsets of R^+, and let $A = (X,S)$ and $B = (Y,P)$ be two approximation spaces, where S and P are equivalence relations on X and Y, respectively, $f : X \to Y$ is a function. Then we define (S,P)-lower approximation of f as the function $f_* : X \to Y$, such that $f_*(x) = P_*\left[f(x)\right]$ for every $x \in X$, and (S, P)-upper approximation of f as the function $f^* : X \to Y$, such that $f^*(x) = P^*\left[f(x)\right]$, for every $x \in X$.

We see that the term $\left(S,P\right)$ in the above definition can be replaced by P only since the approximations of the function f depends only on P.

Let $f : X \to Y$ be a real valued function, where X and Y are two subsets of R^+. The function f is called a rough function at a point $x \in X$ if and only if $f_*(x) \neq f^*(x)$ and f is called a rough function on X if it is a rough function at every point $x \in X$.

We give the following example to indicate the above notions.

***Example* 2.2:** Let $f : R^+ \to R^+$ be a real valued function defined by $f(x)=x+1$ for every $x\in R^+$. We denote the odd and even integers by *O* and *E*, respectively, then $A=\left(R^+,\pi^+(O)\right)$ and $B=\left(R^+,\pi^*(E)\right)$ are approximation spaces, where $\pi(O)$ and $\pi(E)$ are partitions of R^+ defined by $\pi(O)=\{0,(0,1),1,(1,3),\cdots\}$ and $\pi(E)=\{0,(0,1),1,(1,3),\cdots\}$, then at every point $x\in R^+$ we define *E*-lower approximation of *f* by $f_*:R^+\to R^+$ such that

$$f_*=E_*\left[f(x)\right]=E_*\left[x+1\right]=E_*\left[0,x+1\right]$$
$$=\left\{y\in R^+:E(y)\subseteq\left[0,x+1\right]\right\}$$

and the *E*-upper approximation of *f* by the function $f^*:R^+\to R^+$ such that

$$f_*(x)=E^*\left[f(x)\right]=\left\{y\in R^+:E(y)\cap[0,x+1]\neq\varphi\right\}.$$

For $x=3$, we have $f(3)=3+1=4$, then

$$f_*(3)=E_*(4)=E_*\left([0,4]\right)$$
$$=\left\{y\in R^+:E(y)\subseteq[0,4]\right\}=[0,4]$$
$$=\left\{y\in R^+:E(y)\cap[0,4]\neq\varphi\right\}=[0,4].$$

and $f^*(3)=E^*(4)=E^*\left([0,4]\right)$. Then *f* is an exact function at $x=3$, similarly we can prove that *f* is an exact functional at every odd natural number.

For $x=2$, then

$$f_*(2)=E_*(3)=E_*\left([0,3]\right)$$
$$=\left\{y\in R^+:E(y)\subseteq[0,3]\right\}=[0,2]$$

But

$$f^*(2)=E^*(3)=E^*\left([0,3]\right)$$
$$=\left\{y\in R^+:E(y)\cap[0,3]\neq\varphi\right\}=[0,4]$$

Then f is a rough function at $x=2$, similarly it can be proved that f is a rough function at every even natural number.

Also, we notice that f is a rough function at every $x\notin N$. Then f is a rough function at every point $x\notin N$ or x is an even natural number.

Let $f:X\to Y$ be a real valued function. Then f is called (S,P)-continuous (roughly continuous) at a point $x\in X$ if $f\left[clS(x)\right]\subseteq P^*\left(\left[f(x)\right]\right)$, where $A=(X,S)$ and $B=(Y,P)$ are approximation spaces.

Let $f:X\to Y$ be a real valued function. Then f is roughly continuous on X if f is a roughly continuous at every point $x\in X$.

***Example* 2.3:** According to Example 2, the function $f:R^+\to R^+$ is a rough function at $x=1.5$ but $f\left(clS(1.5)\right)=[2,4]$ and $P^*\left(f(1.5)\right)=E^*[2.5]=[0,4)$, then *f* is not a rough continuous function at the rough number $x=1.5$, but at $x=1$, since $f\left(clS(1)\right)=\{2\}$ and $E^*\left(f(1)\right)=[0,2]$ then *f* is a roughly continuous at $x=1$, also at every $x\in N$ such that *x* is odd number *f* is roughly continuous. If *x* is an even number, then *f* is not a roughly continuous; hence *f* is not a roughly continuous function on R^+.

***Example* 2.4:** Let *X* and *Y* be subsets of R^+, such that $X=\{1,3,5,7\}$ and $Y=\{2,4,6\}$ and the real valued function $f:X\to Y$ be defined by $f(1)=f(5)=2$, $f(3)=4$ and $f(7)=6$, and consider the approximation spaces $A=(X,S)$ and $B=(Y,P)$, where $X\setminus S=\{\{1,5\},\{3,7\}\}$ and $Y\setminus P=\{\{2\},\{4,6\}\}$ we define the function $f_*:X\to Y$ by $f_*(x)=P_*\left(f(x)\right)$. Then,

$f_*(1)=P_*(2)=\{2\}$, $f_*(3)=P_*(4)=\varphi$,
$f_*(5)=P_*(2)=\{2\}$, $f_*(7)=P_*(6)=\varphi$.

Also for the function $f^*:X\to Y$ such that $f^*(x)=P^*\left(f(x)\right)$. Then, $f^*(1)=P^*(2)=\{2\}$, $f^*(3)=P^*(4)=\{4,6\}$, $f^*(5)=P^*(2)=\{2\}$, $f^*(7)=P^*(6)=\{4,6\}$. Then the function *f* is *P*-rough at $x=3.7$ and *f* is not *P*-rough function at $x=1.5$.

Now, if $x=1$, then $Seq(x)=clSeq(x)=\{1,5\}$ and we have $f\left(clSeq(x)\right)=f\left(\{1,5\}\right)=\{2\}$ and $P^*\left(f(x)\right)=P^*\left(\{2\}\right)=\{2\}$, then $f\left(clSeq(x)\right)\subseteq P^*\left(f(x)\right)$, *i.e.*, the function *f* is (S,P)-roughly continuous at $x=1$.

If $x=3$, then $Seq(x)=clSeq(x)=\{3,7\}$ and $f\left(\{3,7\}\right)=\{4,6\}$, but $P^*\left(f(x)\right)=P^*\left(\{4\}\right)=\{4,6\}$ then, the function *f* is (S,P)-roughly continuous at $x=3$. Also at $x=5.7$ we find that *f* is (S,P)-roughly continuous, hence *f* is (S,P)-roughly continuous on *X*.

3. Topological Pre-Rough Functions

We purpose to generalize the concept of rough function to topological pre-rough function by using pre-open sets in topological spaces. Let (X,τ) be a topological space and $x\in X$. Then $pmin(x)=\bigcap\left\{G\in PO(X):x\in G\right\}$ is called the pre minimal set containing the point x with respect to pre-open sets in the topology τ on *X*.

The principle topology on a set *X* is the topology has the minimal bases that consists only of minimal open sets at each $x\in X$.

***Theorem* 3.1:** A topology τ on a set *X* is principle iff arbitrary intersections of members of τ are members of τ [20].

Let (X,τ) be a principle topological space, for any element $x\in X$, we define pre-sequence by the set $Seq(x)=\bigcap\left\{G:G\in PO(X),x\in G\right\}$ and by ${}^{p}\overline{Seq}(x)$ we mean the pre-closure of $Seq(x)$ in (X,τ).

If $f:(X,\tau)\to(Y,\sigma)$ is a function between principle spaces (X,τ) and (Y,σ), we define the functions $pmin_f:(X,\tau)\to(Y,\sigma)$, by $pmin_f(x)=\bigcap\{G':G'\in PO(Y)$ and $f(x)\in G'$ for every $\{x\in X\}$, and $pcl_f:(X,\tau)\to(Y,\sigma)$, by $pcl_f(x)=pcl_\sigma\left(f(x)\right)$ for all $x\in X$.

Let $f:(X,\tau)\to(Y,\sigma)$ be a function, where *X* and *Y* are principle spaces. The function *f* is called a topological pre-rough function at the point *x* in *X* if and only if $pmin_f(x)\neq pcl_f(x)$, and *f* is a topological pre-rough function on *X* if it is a topological pre-rough function at every point *x* in *X*.

***Example* 3.1:** Let (X,τ) and (Y,σ) be topological spaces, where $X=\{a,b,c,d\}$,
$\tau=\{X,\varphi,\{a\},\{c\},\{d\},\{a,c\},\{a,d\},\{c,d\},\{a,c,d\}\}$
and $Y=\{1,2,3,4\}$, $\sigma=\{Y,\varphi,\{1\},\{2\},\{1,2\},\{1,2,3\}\}$. Let $f:(X,\tau)\to(Y,\sigma)$ be a map defined by $f(a)=3$, $f(b)=1$, $f(c)=4$ and $f(d)=2$. We have the following table (**Table 1**).

Consequently, $pmin_f(x)\neq pcl_f(x)$ for every $x\in X$, hence *f* is a topological pre-rough function on X.

A function $f:(X,\tau)\to(Y,\sigma)$ is said to be a topological pre-rough continuous at the point $x\in X$ if and only if $f\left({}^{p}\overline{Seq}(x)\right)\subseteq pcl_\sigma\left(Seq(f(x))\right)$, and it is a topological pre-rough continuous on X if it is a topological pre-rough continuous at every point $x\in X$.

***Example* 3.2:** Let (X,τ) and (Y,σ) be topological spaces, where $X=\{a,b,c,d\}$ and $Y=\{1,2,3,4\}$ with $\tau=\{X,\varphi,\{a\},\{b\},\{a,b\},\{a,b,c\}\}$ and
$\sigma=\{Y,\varphi,\{1\},\{1,4\},\{1,2,4\}\}$. Let $f:(X,\tau)\to(Y,\sigma)$ be a map defined by $f(a)=1$, $f(b)=2$, $f(c)=4$ and $f(d)=3$ (**Table 2**).

Consequently, $f\left({}^{p}\overline{Seq}(x)\right)\subset pcl_\sigma\left(Seq(f(x))\right)$ for every $x\in X$, hence *f* is a topological pre-rough continuous function on *X*.

4. The Pre-Approximations of Functions

A function *f* from *X* to Y is a relation from X to Y that satisfies:

1) $Dom(f)=X$.

2) If $(x,y)\in f$ and $(x,z)\in f$, then $y=z$.

If $X=Y$, we say f is a function on X. A function $f:X\to Y$ is completely represented by its graph $g(f)=\{(x,f(x)):x\in X\}$.

The concept of rough relations is defined by using a certain type of relation products. The following proposition will simplify the process of getting $(U_1\times U_2)/(R_1\times R_2)$ via (U_1/R_1) and (U_2/R_2).

***Theorem* 4.1:** Let $A_1=(U_1,R_1)$ and $A_2=(U_2,R_2)$ be a pre-approximation spaces. Then we have $(U_1\times U_2)/(R_1\times R_2)=(U_1/R_1)\times(U_2/R_2)$.

***Proof*:** Since for any $u,u'\in U_1$ and $v,v'\in U_2$, we have, $((u,v),(u',v'))\in R_1.R_2$ iff $(u,u')\in R_1$ and $(v,v')\in R_2$. Let $[(u,v)]_{R_1\cdot R_2}\in(U_1\times U_2)/(R_1\times R_2)$. Then we have

$$\begin{aligned}[(u,v)]_{R_1\cdot R_2}&=\{(u',v'):((u,v),(u',v'))\in R_1.R_2\}\\&=\{(u',v'):(u,u')\in R_1\text{ and }(v,v')\in R_2\}\\&=\{u':(u,u')\in R_1\}\times\{v':(v,v')\in R_2\}\\&=[u]_{R_1}\times[v]_{R_2}\end{aligned}$$

Hence $(U_1\times U_2)/(R_1\times R_2)=(U_1/R_1)\times(U_2/R_2)$.

Let $f:(U_1,R_1)\to(U_2,R_2)$ be any function, where $A_1=(U_1,R_1)$ and $A_2=(U_2,R_2)$ are pre-approximation spaces, such that R_1 and R_2 are equivalence relations on U_1 and U_2 respectively. We define the equivalence relation $R=R_1\times R_2$ such that $(U_1\times U_2)/R=(U_1/R_1)\times(U_2/R_2)$ is a partition of $U_1\times U_2$ for the function $g(f)=\{(x,f(x)):x\in U_1\}$ we define the pre-approximations

$${}_p\underline{R}(g(f))=\{(u_1,u_2)\in U_1\times U_2:[(u_1,u_2)]_R\subseteq g(f)\}$$

$${}_p\underline{R}(g(f))=\{(u_1,u_2)\in U_1\times U_2:[(u_1,u_2)]_R\cap g(f)\neq\varphi\}$$

A function $f:\ U_1\to U_2$ is said to be roughly in the pre-approximation space $A=(U^2,\ R)$, where $A_1=(U1,\ R1)$ and $A_2=(U_2,\ R_2)$ are pre-approximation spaces and $A=A_1\times A_2$, $U^2=U_1\times U_2$ if

$${}_p\underline{R}(g(f))\neq{}^{p}\overline{R}(g(f)),$$

otherwise *f* is pre-exact function.

***Example* 4.1:** Let $U1=\{a,b,c,d,e\}$ and $U_2=\{1,2,3,4,5,6\}$ and consider the function $f:U_1\to U_2$ defined by $g(f)=\{(a,3),(b,6),(c,5),(d,2),(e,3)\}$. Consider the partitions $U_1/R_1=\{\{a,c\},\{b\},\{d,e\}\}$ and $U_2/R_2=\{\{1\},\{2,5\},\{3,4\},\{6\}\}$. Then

$$(U_1\times U_2)/R=(U_1/R_1)\times(U_2/R_2)$$

$$=\left\{\begin{array}{l}\{(a,1),(c,1)\},\{(a,2),(c,2),(a,5),(c,5)\},\\\{(a,3),(a,4),(c,3),(c,4)\},\{(a,6),(c,6)\},\\\{(b,1)\},\{(b,2),(b,5)\},\{(b,3),(b,4)\},\{(b,6)\},\\\{(d,1),(e,1)\},\{(d,2),(d,5),(e,2),(e,5)\},\\\{(d,3),(d,4),(e,3),(e,4)\},\{(d,6),(e,6)\}\end{array}\right\}$$

is a partition of $(U_1\times U_2)$.

Table 1. $pmin_f(x)$ and $pcl_f(x)$ for some subsets of X.

X	$pmin_f(x)$	$pcl_f(x)$
{a}	{1, 2, 3}	{3}
{b}	{1}	{1, 3, 4}
{c}	Y	{4}
{d}	{2}	{2, 3, 4}

Table 2. Topological pre-rough continuous function on *X*.

X	*Seq*(x)	${}^{p}\overline{Seq}(x)$	$f\left({}^{p}\overline{Seq}(x)\right)$	$pcl_\sigma\left(Seq(f(x))\right)$
{a}	{a}	{a, c, d}	{1, 3, 4}	Y
{b}	{b}	{b, c, d}	{2, 3, 4}	Y
{c}	{a, b, c}	X	Y	Y
{d}	{a, b, d}	X	Y	Y

Then $p\underline{R}(g(f))=\{(b,6)\}$ and

$$^{p}\overline{R}(g(f))=\begin{Bmatrix}(a,3),(a,4),(c,3),(c,4),(b,6),(a,2),\\(c,2),(a,5),(c,5),(d,2),(d,5),(e,2),\\(e,5),(d,3),(d,4),(e,3),(e,4)\end{Bmatrix}$$

Therefore the function f is a rough function such that $_{p}\underline{R}(g(f))\neq{}^{p}\overline{R}(g(f))$.

For the function $f:U_1\to U_2$, we observe that in general $^{p}\overline{R}(g(f))$ and $_{p}\underline{R}(g(f))$ are not functions from $U1$ into $U2$. We point that, the process of defining an pre-approximations on $(U1\times U2)$ such that $^{p}\overline{R}(g(f))$ and $_{p}\underline{R}(g(f))$ are functions is an open question to be solved in our next work.

***Theorem* 4.2:** For every function $f:U_1\to U_2$ such that $A_1=(U_1,R_1)$ and $A_2=(U_2,R_2)$ are selective pre-approximation spaces then f is an pre-exact function.

***Proof*:** Since in any selective pre-approximation space, $[(x,y)]_R=\{(x,y)\}$ then $^{p}\overline{R}(g(f))={}_{p}\underline{R}(g(f))$ then f is an preexact function.

***Example* 4.2:** Let $U_1=\{a,b,c,d\}$ and $U2=\{1,2,3\}$. Consider the function $f:U1\to U2$, defined by $g(f)$ $=\{(a,1),(b,1),(c,2),(d,3)\}$ and consider the partitions $U_1/R_1=\{\{a\},\{b\},\{c\},\{d\}\}$ and $U_2/R_2=\{\{1\},\{2\},\{3\}\}$. Then

$$U_1\times U_2/R=(U_1/R_1)\times(U_2/R_2)=$$

$$\begin{Bmatrix}\{(a,1)\},\{(a,2)\},\{(a,3)\},\{(b,1)\},\{(b,2)\},\{(b,3)\},\\\{(c,1)\},\{(c,2)\},\{(c,3)\},\{(d,1)\},\{(d,2)\},\{(d,3)\}\end{Bmatrix}$$

is a partition of $U1\times U2$.

Then $_{p}\underline{R}(g(f))=\{(a,1),(b,1),(c,2),(d,3)\}$ and $^{p}\overline{R}(g(f))=\{(a,1),(b,1),(c,2),(d,3)\}$, then f is an pre-exact function.

For a function $f:U1\to U2$ such that $A_1=(U1,R1)$ and $A_2=(U_2,R_2)$ are selective pre-approximation spaces then

1) If f is a one-to-one function then also both $^{p}\overline{R}(g(f))$ and $_{p}\underline{R}(g(f))$.

2) If f is onto function then also both $^{p}\overline{R}(g(f))$ and $_{p}\underline{R}(g(f))$.

3) If f is a pre-continuous function then also both $^{p}\overline{R}(g(f))$ and $_{p}\underline{R}(g(f))$.

No function $f:U_1\to U_2$ such that $A_1=(U_1,R_1)$ and $A_2=(U_2,R_2)$ are not selective approximation spaces is pre-exact, and f is not a constant function.

5. An Alternative Description of Topological Pre-Rough Functions

Let (U_1,τ_1) and (U_2,τ_2) be any topological spaces, the function $f:(U1,\tau1)\to(U_2,\tau_2)$, can be considered as a relation of $U_1\times U_2$ and if β_1 is a basis of τ_1 and β_2 is a basis of τ_2, then $\beta=\beta_1\times\beta_2$ is a basis of the topology τ on $U_1\times U_2$. In the topology $(U_1\times U_2,\tau)$ we define $_{p}\underline{f}=pint(f)$ and $^{p}\overline{f}=pcl(f)$ for the function f. Let $f:(U_1,\tau_1)\to(U_2,\tau_2)$ be a function, where (U_1,τ_1) and (U_2,τ_2), are topological spaces, the function f is called a topological pre-rough function in $(U_1\times U_2,\tau)$ iff $_{p}\underline{f}\neq{}^{p}\overline{f}$ otherwise, f is an preexact function in $(U_1\times U_2,\tau)$.

***Example* 5.1:** Let (U_1,τ_1) and (U_2,τ_2) be any topological spaces where $U_1=(a,b,c)$,

$U_2=\{1,2,3,4\}$, $\tau_1=\{U_1,\varphi,\{a\},\{b,c,d\}\}$,

$\tau_2=\{U_2,\varphi,\{3\},\{1,2,4\}\}$ Consider $\beta_1=\{\{a\},\{b,c,d\}\}$ and $\beta_2=\{\{3\},\{1,2,4\}\}$ are basis of τ_1 and τ_2 respectively. Let $f:U_1\to U_2$, $g:U_1\to U_2$ and $h:U_1\to U_2$ are mappings defined by

$f=\{(a,3),(b,1),(c,2),(d,4)\}$,

$g=\{(a,2),(b,3),(c,1),(d,4)\}$

and $h=\{(a,3),(b,3),(c,3),(d,3)\}$.

Then $_{p}\underline{f}=pinf(f)=\{(a,3)\}$ and

$$^{p}\overline{f}=cl(f)=\begin{Bmatrix}(a,3),(b,1),(b,2),(b,4),(c,1),\\(c,2),(c,4),(d,1),(d,2),(d,4)\end{Bmatrix}$$

Then f is a pre-rough function in $(U_1\times U_2,\tau)$. Also, $_{p}\underline{g}=pint(g)=\varphi$ and $^{p}\overline{g}=pcl(g)=U_1\times U_2$

We call g is a function not defined from pre-lower and from upper. Finally, for the constant function h, we have $_{p}\underline{h}=pint(h)=pcl(h)={}^{p}\overline{h}$, and h is an pre-exact function. In fact, h is the only exact function in $(U_1\times U_2,\tau)$.

According to Example 1, we have the following:

1) The function f is continuous, but $_{p}\underline{f}$ and $^{p}\overline{f}$ are not functioning, hence we cannot say that $\underline{f}$ or $\overline{f}$ is pre-continuous.

2) The function h is always precontinuous function, and it is an pre-exact function, hence $_{p}\underline{h}$ and $^{p}\overline{h}$ is pre-continuous functions.

6. Experiments and Evaluations

This section shows the effectiveness of using pre-rough functions for extracting new data from multi-valued information systems.

In this section, we briefly describe the Rheumatic Fever datasets mentioned in [37] as a topological application of rough functions. As mentioned in [39] rheumatic fever is a very common disease and it has many symptoms differs from patient to another though the diagnosis is the same. So, we obtained the following example on four rheumatic fever patients. All patients are between 9-12 years old with a history of Arthurian began from age 3-5 years. This disease has many symptoms and it is usually started in young age and still with the patient along his life.

Table 3 in [37] introduced the seven patients characterized by 8 symptoms (attributes) using them to decide the diagnosis for each patient (decision attribute). Where the attributes are satisfied in **Table 2** in [37].

We recall and sell it here **Table 3**.

If we defined the following mapping on **Table 3**:

$f: U \to P(U)$:

$f(p1) = \{p1, p2\}, f(p2) = \{p2, p3\}$,

$f(p3) = \{p3\}, f(p4) = \{p2, p4\}, f(p5) = \{p1, p5, p7\}$

$f(p6) = \{p6\}, f(p7) = \{p5, p7\}$

From the relation $R_a = \{(x, y) : f_a(x) \subseteq f_a(y)\}$ where a is an element of the power set of the set of condition attributes $\{\alpha, \beta, \delta\}$. The the following classes $\chi 1 = \{xR_a : x \in U\}$ and $\chi 2 = \{R_a x : x \in U\}$ are two subbases of two topologies on U such that $R_a x = \{y : yR_a x\}$. Then according to **Table 3** we have the following couples of topologies:

$$\tau_1^{\alpha} = \left\{\begin{array}{l} U, \varphi, \{p_2\}, \{p_3\}, \{p_2, p_3\}, \{p_1, p_2\}, \\ \{p_1, p_2, p_3\}, \{p_2, p_3, p_4, p_5, p_6, p_7\}, \\ \{p_1, p_2, p_4, p_5, p_6, p_7\}, \{p_2, p_4, p_5, p_6, p_7\} \end{array}\right\},$$

$$\tau_2^{\alpha} = \left\{\begin{array}{l} U, \varphi, \{p_1\}, \{p_3\}, \{p_1, p_3\}, \{p_4, p_5, p_6, p_7\}, \\ \{p_3, p_4, p_5, p_6, p_7\}, \{p_1, p_4, p_5, p_6, p_7\}, \\ \{p_1, p_2, p_4, p_5, p_6, p_7\}, \{p_2, p_3, p_4, p_5, p_6, p_7\} \end{array}\right\}$$

$$\tau_1^{\beta} = \left\{\begin{array}{l} U, \varphi, \{p_5\}, \{p_7\}, \{p_3, p_7\}, \{p_1, p_4\}, \\ \{p_5, p_7\}, \{p_3, p_5, p_7\}, \{p_1, p_4, p_5\}, \\ \{p_1, p_4, p_5, p_7\}, \{p_1, p_2, p_4, p_6\} \end{array}\right\}$$

$$\tau_2^{\beta} = \left\{\begin{array}{l} U, \varphi, \{p_3, p_5, p_7\}, \{p_2, p_3, p_6\}, \{p_2, p_3, p_6, p_7\}, \\ \{p_1, p_2, p_4, p_6\}, \{p_1, p_2, p_3, p_4, p_6\}, \\ \{p_2, p_3, p_5, p_6, p_7\}, \{p_1, p_2, p_4, p_5, p_6\}, \\ \{p_1, p_2, p_3, p_4, p_5, p_6\}, \{p_1, p_2, p_3, p_4, p_6, p_7\} \end{array}\right\}$$

$$\tau_1^{\delta} = \left\{\begin{array}{l} U, \varphi, \{p_4\}, \{p_5\}, \{p_2, p_7\}, \{p_4, p_5\}, \\ \{p_2, p_4, p_7\}, \{p_2, p_5, p_7\}, \{p_2, p_4, p_5, p_7\}, \\ \{p_1, p_2, p_3, p_5, p_6, p_7\} \end{array}\right\}$$

$$\tau_2^{\delta} = \left\{\begin{array}{l} U, \varphi, \{p_4\}, \{p_1, p_3, p_6\}, \{p_1, p_3, p_4, p_6\}, \\ \{p_1, p_3, p_5, p_6\}, \{p_1, p_2, p_3, p_6, p_7\}, \\ \{p_1, p_2, p_3, p_5, p_6, p_7\}, \{p_1, p_3, p_4, p_5, p_6\}, \\ \{p_1, p_2, p_3, p_4, p_6, p_7\} \end{array}\right\}$$

$\tau_1^{\alpha\beta} = \tau_1^{\alpha} \cap \tau_1^{\beta} = \{U, \varphi\}$

$\tau_2^{\alpha\beta} = \tau_2^{\alpha} \cap \tau_2^{\beta} = \{U, \varphi\}$

$\tau_1^{\alpha\delta} = \tau_1^{\alpha} \cap \tau_1^{\delta} = \{U, \varphi\}$

$\tau_2^{\alpha\delta} = \tau_2^{\alpha} \cap \tau_2^{\delta} = \{U, \varphi\}$

$\tau_1^{\beta\delta} = \tau_1^{\beta} \cap \tau_1^{\delta} = \{U, \varphi, \{p_5\}\}$

$\tau_2^{\beta\delta} = \tau_2^{\beta} \cap \tau_2^{\delta} = \{U, \varphi, \{p_1, p_2, p_3, p_4, p_6, p_7\}\}$

$\tau_1^{\alpha\beta\delta} = \tau_1^{\alpha} \cap \tau_1^{\beta} \cap \tau_1^{\delta} = \{U, \varphi\}$

$\tau_2^{\alpha\beta\delta} = \tau_2^{\alpha} \cap \tau_2^{\beta} \cap \tau_2^{\delta} = \{U, \varphi\}$

According to the mapping $f: U \to P(U)$ and using each one of the above topologies we can deduce that the decision topology can be given by:

$\tau_D = \{U, \phi, \{p_1, p_2, p_3, p_6, p_7\}, \{p_4\}, \{p_5\}, \{p_4, p_5\}$.

Now we can construct a familiar system of **Table 3** contains only the pre-rough images constructed using the terminology of pre-rough functions. This system can be the reduction system of **Table 3** and it given in **Table 4**.

This means that we can remove the conditional attribute $\{\alpha\}$ without any loss of information.

7. Conclusions

We conclude that the emergence of topology and its operators [38,39] in the construction of some rough set concepts will help to get rich results that yields a lot of logical statements which discover hidden relations between data and moreover, probably help in producing

Table 3. Multi-valued information system of [37].

D	δ	β	α	U
p1	$\{\alpha_2\}$	$\{\beta_1, \beta_2, \beta_4\}$	$\{\delta_1\}$	$\{d_3\}$
p2	$\{\alpha_1, \alpha_2\}$	$\{\beta_1, \beta_2\}$	$\{\delta_1, \delta_3\}$	$\{d_3\}$
p3	$\{\alpha_3\}$	$\{\beta_1, \beta_3\}$	$\{\delta_1\}$	$\{d_3\}$
p4	$\{\alpha_1\}$	$\{\beta_1, \beta_2, \beta_4\}$	$\{\delta_4\}$	$\{d_1\}$
p5	$\{\alpha_1\}$	$\{\beta_5\}$	$\{\delta_1, \delta_2\}$	$\{d_2\}$
p6	$\{\alpha_1\}$	$\{\beta_1, \beta_2\}$	$\{\delta_1\}$	$\{d_3\}$
p7	$\{\alpha_1\}$	$\{\beta_1, \beta_3, \beta_4\}$	$\{\delta_1, \delta_3\}$	$\{d_3\}$

Table 4. Reduced System.

U	β	δ
p1	$\{\beta_1, \beta_2, \beta_4\}$	$\{\delta_1\}$
p2	$\{\beta_1, \beta_2\}$	$\{\delta_1, \delta_3\}$
p3	$\{\beta_1, \beta_3\}$	$\{\delta_1\}$
p4	$\{\beta_1, \beta_2, \beta_4\}$	$\{\delta_4\}$
p5	$\{\beta_5\}$	$\{\delta_1, \delta_2\}$
p6	$\{\beta_1, \beta_2\}$	$\{\delta_1\}$
P7	$\{\beta_1, \beta_3, \beta_4\}$	$\{\delta_1, \delta_3\}$

accurate programs. These topological operators will play an essential role in data mining and knowledge discovery in databases. In this paper, we give an overview of several dissipated results on the pre-rough functions. More specifically, we attempt to show: usefulness of this new concept in a calculus of rough functions.

The future application of this work will be useful in many fields such as Fuzzy Expert Systems [40] by generalizations of rough functions for fuzzy rough functions. It also is useful in knowledge discovery methods [41].

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